

A NOTE ON JOINTLY MODELING EDGES AND NODE ATTRIBUTES OF A NETWORK

HAIYAN CAI

ABSTRACT. We are interested in modeling networks in which the connectivity among the nodes and node attributes are random variables and interact with each other. We propose a probabilistic model that allows one to formulate jointly a probability distribution for these variables. This model can be described as a combination of a latent space model and a Gaussian graphical model: given the node variables, the edges will follow independent logistic distributions, with the node variables as covariates; given edges, the node variables will be distributed jointly as multivariate Gaussian, with their conditional covariance matrix depending on the graph induced by the edges. We will present some basic properties of this model, including a connection between this model and a dynamical network process involving both edges and node variables, the marginal distribution of the model for edges as a random graph model, its one-edge conditional distributions, the FKG inequality, and the existence of a limiting distribution for the edges in a infinite graph.

1. INTRODUCTION

In modeling networks ([13, 15, 20]), the usual focus is on the network topologies, or the configurations of edges. A network is typically modeled as a random graph ([3, 6, 8]) defined in terms of a probability distribution of the edge status. However, networks in many applied problems are not always just about links or edges. More extensive data for certain networks, containing information not only for edges but also for some node variables or attributes, are becoming available. For such data and for some important problems in network study, a limitation of the random graph model is the absence of information from the node variables. Such a model is incapable of catching interactions between edges and nodes. In a social network problem, for example, one might be interested in studying users' behaviors (or some dynamical attribute in the user-profiles) as a function of the network topology, or vice versa ([18, 19]), or in a gene network problem, one might be interested in inferring gene expression

2000 *Mathematics Subject Classification.* 60, 62, 82.

Key words and phrases. Network Model; Dynamical Network; Random Graph; Limiting Graph.

levels as a function of an underlying regulatory network, or vice versa ([22]). When it comes to analyzing behaviors of the nodes in a network or the influence of node behaviors on network topologies, the utility of the random graph models becomes limited.

The latent space model ([9, 11, 15]) is another popular network model. It does assume the dependence of the edge probabilities on some node variables. The model however treats these variables as latent variables, paying little attention to the inference on these variables. On the other hand, a Gaussian graphical model describes a distribution for node variables on a network with built-in edge information of the network (through the inverse covariance matrix). It however treats the network topology as a static parameter which remains constant regardless how the node variables will change.

In this paper, we propose a joint probability distribution for both edges and node variables (Section 2). A study of such a model can shed lights on how edges and nodes interact with each other in a network so that information for both edges and nodes can be utilized in studying the networks. In a way, our model can be described as a combination of the latent space model and the Gaussian graphical model: given the node variables, the edges will follow independent logistic distributions, with the node variables as covariates in the logistic function; given edges, the node variables will be distributed jointly as multivariate Gaussian, with their conditional covariance matrix depending on the graph induced by the edges. In terms of the marginal distribution for the node variables, our model generalizes the Gaussian graphical model to allow for the underlying graphical structure to be random. In other words, it is now a mixture of Gaussian graphical models over all the possible edge configurations of the network, and the weights in this mixture are the probabilities of the corresponding network configurations. Our model also leads to a non-trivial and interesting random graphical model when we take the marginal distribution for the edges in our model. This graphical model is different from all the other models that we have been in the literature so far. As we will see, the probability of a network configuration under this random graphical model is proportional to the square root of the determinant of the corresponding conditional covariance matrix of the node variables.

A reason that motivates us to propose such a model is that it provides a sensible framework for modeling network dynamics in which the edge status and the node variables change their values over time, as we will explain in the end of Section 2. We will see that the dynamical

system updates edge status and node variables alternatively according to the conditional distributions between edges and nodes. The equilibrium (stable) distribution of the dynamical system is then exactly the joint distribution we propose here. In other words, our model can be viewed as the stable probability law of a dynamical network process. For data containing both edges and node variables, our model can be fairly easy to fit, because of the simple forms of the conditional probabilities. If only edge data are available, techniques developed for the latent space models may be adopted. We will not discuss these issues any further here. This paper is mainly about basic probabilistic properties of the model.

We will pay particular attention to the marginal distribution for edges of our model. An explicit formula for the conditional probability of one edge given all other edges is given in Section 3. The formula has a simple form and can be useful for predicting one edge's status based on observations from other edges. We will show in Section 4 that the probability distribution for edges is positively associated in the sense that it satisfies the FKG inequality ([14]), a property that is shared by many well-known models in statistical mechanics. We then give a weak convergence result for the edge distribution based on the FKG inequality. To ensure consistent results in statistical analysis for very large networks, it is essential that the model, as a probability law, has a limiting distribution. The concept of the limit for random graphs we use here is that of the infinite-volume Gibbs distributions on graphs, involving both nodes and edges (see [14] for an example). Our approach does not depend on the concept of metrics for graphs and therefore is different from those seen in another line of research on graph limits ([4, 5, 7, 17]). We also note that there is a close similarity between our model and the random-cluster model derived in the statistical mechanics ([14]): what our model is to the Gaussian graphical model is in some sense similar to what the random-cluster model is to Ising or Potts models. This is indeed another reason that motivated us to propose the model in this note.

2. THE RANDOM GAUSSIAN GRAPHICAL MODEL

We will call our model the random Gaussian graphical model and formulate it in this section. Let $G = (V, E)$ be a finite simple graph (undirected, unweighted, no loops, no multiple edges) with E being a subset of $V \times V$ which is fixed. Suppose $|V| = m$ and $|E| = n$. For convenience we identify V as the integer set $V = \{1, \dots, m\}$. Suppose associated with each node $i \in V$

there is a random variable X_i , representing an attribute of node i . Let $X = \{X_1, \dots, X_m\}$. We will use $x \in \mathcal{R}^m$ to denote a generic value of X . We write (i, j) for the edge in E which is incident with the nodes $i, j \in V$.

We will consider random sub-graphs of G in which V remains the same and E is reduced randomly to some subset of itself. Such a random graph can be represented by a random adjacency matrix $A = \{A_{ij}, i, j \in V\}$ in which all the diagonal elements $A_{ii} = 0$, and for each edge $(i, j) \in E$, $A_{ij} = A_{ji} = 1$ if the edge is present in the random graph, and $A_{ij} = A_{ji} = 0$ if otherwise. It is always understood that $A_{ij} \equiv 0$ for all $(i, j) \notin E$. We will call A_{ij} an edge variable. With a slight abuse of notation, we let $\mathcal{A} = \{0, 1\}^E$ be the set of all possible values of A . We will use $a = a^T \in \mathcal{A}$ to denote a generic value of the adjacency matrix A .

By “random Gaussian graphical model” we mean the following joint probability density for variables A and X , defined on the space $\mathcal{A} \times \mathcal{R}^m$,

$$(1) \quad \mu(a, x) \equiv \frac{1}{Z} \exp \left\{ -\frac{1}{2} H(a, x) \right\}, \quad (a, x) \in \mathcal{A} \times \mathcal{R}^m,$$

where

$$(2) \quad H(a, x) = \alpha \sum_i x_i^2 + \beta \sum_{(i,j) \in E} a_{ij} (x_i - x_j)^2$$

for some parameters $\alpha > 0$ and $\beta \geq 0$, and

$$Z = \sum_{a \in \mathcal{A}} \int_{\mathcal{R}^m} \mu(a, x) dx$$

is the normalizing constant. It is clear that this Z is always finite.

We note that in this model, if all $a_{ij} = 1$ it becomes an usual Gaussian graphical model (as we will see below). Therefore we can consider the Gaussian graphical model as a “full model” relative to the given edge set E while model (1) as a model that allows us to “turn off” some edges in E at random (and therefore remove the associated correlation terms among the nodes in (2)) according to the values of a_{ij} ’s. Model (1) is like a Gaussian graphical model on a less connected graph obtained by removing some edges from E randomly, and therefore the name “random Gaussian graphical model”. In particular, if all $a_{ij} = 0$ and therefore there are no connections among the nodes in the graph, X_i ’s are i.i.d. $N(0, 1/\alpha)$ random variables. The joint distribution of values of a_{ij} ’s are in turn depended on X_i ’s. In general, the likelihood of

connectivity among the nodes is determined by the magnitudes of the differences between the corresponding node variables and the value of β . On the other hand, the connectivity of the nodes will, in turn, affect the distribution of the node variables.

To study this model, it is more convenient to rewrite $\mu(a, x)$ in a matrix form as follows. Let e_i be an m -dimensional column vector such that its i th element is 1 and all others are 0. We define a matrix, as a function of $A = a$,

$$(3) \quad Q(a) = \alpha I + \beta \sum_{(i,j) \in E} a_{ij} (e_i - e_j)(e_i - e_j)^T, \quad a \in \mathcal{A}.$$

Then

$$H(a, x) = x^T Q(a) x.$$

Note that the term

$$L(a) = \sum_{(i,j) \in E} a_{ij} (e_i - e_j)(e_i - e_j)^T$$

in (3) is the graphical Laplacian of the graph for $A = a$. Also note that since $H(a, x) > 0$ for all $x \neq 0$ and all $a \in \mathcal{A}$, $Q(a)$ is positive definite for all $a \in \mathcal{A}$. Now let

$$\Sigma(a) = Q(a)^{-1}.$$

Then

$$\begin{aligned} Z &= \sum_a \int_{\mathcal{R}^m} \exp \left\{ -\frac{1}{2} x^T Q(a) x \right\} dx \\ &= \sum_a |2\pi \Sigma(a)|^{1/2}, \end{aligned}$$

where $|2\pi \Sigma(a)|$ is the determinant of the matrix $2\pi \Sigma(a)$. It follows that

$$(4) \quad \mu(a, x) = \frac{|\Sigma(a)|^{1/2}}{\sum_{a'} |\Sigma(a')|^{1/2}} \phi(x|0, \Sigma(a)),$$

where $\phi(x|0, \Sigma(a))$ is the density function of a 0 mean multivariate normal distribution with covariance matrix $\Sigma(a)$, $N(0, \Sigma(a))$.

The marginal distribution of (4) for A provides a model for the random graph and it takes the form:

$$(5) \quad \mu_A(a) = \frac{1}{\kappa} |\Sigma(a)|^{1/2}, \quad a \in \mathcal{A},$$

where κ is the normalizing constant

$$\kappa = \sum_{a'} |\Sigma(a')|^{1/2}.$$

Therefore $\mu_A(a)$ is simply proportional to $|\Sigma(a)|^{1/2}$. The marginal distribution for X can be written as

$$(6) \quad \mu_X(x) = \sum_a \mu_A(a) \phi(x|0, \Sigma(a)), \quad x \in \mathcal{R}^m.$$

This is a mixture of the usual Gaussian graphical models with the precision matrices $Q(a)$ and the corresponding weight $\mu_A(a)$ for each realization of A .

Modeling dynamical networks in which connectivity and node variables are changing in time is an important problem ([16, 21]). The distribution proposed in (4) relates naturally to such a dynamical network process which we can easily describe below.

The conditional distributions of (4) have the following particularly simple forms. Given X , A_{ij} 's in A are independent and the corresponding probabilities take the logistic form:

$$(7) \quad \mu(a|x) = \prod_{(i,j) \in E} \left(\frac{1}{1 + \exp \{\beta(x_i - x_j)^2\}} \right)^{a_{ij}} \left(\frac{\exp \{\beta(x_i - x_j)^2\}}{1 + \exp \{\beta(x_i - x_j)^2\}} \right)^{1-a_{ij}}.$$

A latent space model is thus the conditional distribution $\mu(a|x)$ with x being treated as latent variables. Given $A = a$, X is simply distributed as a multivariate normal:

$$(8) \quad \mu(x|a) = \phi(x|0, \Sigma(a)).$$

This is a centered Gaussian graphical model with the precision matrix $Q(a)$.

We can think of the joint distribution $\mu(a, x)$ as an equilibrium or stable state of the following network process driven by the conditional distributions (7) and (8). Let's denote the process by $\{A^{(t)}, X^{(t)}\}$, $t = 0, 1, 2, \dots$

Initially, the network consists of disconnected nodes with i.i.d. normal random variables in $X^{(0)} = \{X_1^{(0)}, \dots, X_m^{(0)}\}$:

$$A^{(0)} \equiv 0_{n \times n}, \quad X^{(0)} \sim N(0, \alpha^{-1} I_m).$$

Suppose at time t , the current network is in state $\{A^{(t)}, X^{(t)}\}$. The network updates itself after t at some random time points which we assume are independent of all $A^{(u)}$ and $X^{(u)}$, $u \leq t$.

The connectivity of the network is updated so that, independently, some A_{ij} 's have their status switch between 0 and 1 for some $(i, j) \in E$ and others remain unchanged, according to the conditional distribution (7) with the given $X^{(t)}$:

$$A^{(t+1)} \sim \mu(a|X^{(t)}).$$

This change of connectivity then modifies the conditional covariances among the node variables and at some later independent random time points, values of these variables are updated according to (8) with the updated conditional covariance matrix $\Sigma(A^{(t+1)})$:

$$X^{(t+1)} \sim N(0, \Sigma(A^{(t+1)})).$$

The system evolves in time by repeating this updating process.

One can check that the joint distribution of $A^{(t)}$ and $X^{(t)}$ in this process converges to μ as $t \rightarrow \infty$. In fact, the dynamical process we just described is exactly the process of a ‘‘Gibbs sampler’’ in MCMC computations. The simple forms of the conditional probabilities (7) and (8) allow us to simulate μ easily via an MCMC procedure.

3. CONDITIONAL DISTRIBUTION OF ONE EDGE GIVEN OTHERS

We now turn to the marginal distribution of A . For notational simplicity we will often write $\mu(B)$ for $\mu_A(B)$ for any edge event B . This section is about dependence in distribution of one edge on other edges. We will give an explicit formula for this conditional distribution. It shows in Proposition 2 below that this conditional probability depends on other edges only through the conditional variance of the difference of the corresponding node variables. This result allows us to show the FKG inequality and a weak convergence property for our graphical model.

First, some notation. Let us view the network model as a system $\mathcal{S} = (V, E, A, X, \mu)$ with its components defined as in Section 2. Suppose the system \mathcal{S} is such that it does not contain edge (i', j') in E . Suppose the system $\mathcal{S}' = (V, E', A', X, \mu')$ is an augmented version of \mathcal{S} , obtained by adding to \mathcal{S} the edge (i', j') . Note that both systems \mathcal{S} and \mathcal{S}' have the same node set V . Therefore we have $E' = E \cup \{(i', j')\}$,

$$(9) \quad A' = A + A'_{i'j'}(e_{i'}e_{j'}^T + e_{j'}e_{i'}^T),$$

where $A'_{i'j'} \in \{0, 1\}$ is the new edge variable corresponding to the edge (i', j') , and

$$(10) \quad \mathcal{A}' = \{a' : a' = a + a'_{i'j'}(e_{i'}e_{j'}^T + e_{j'}e_{i'}^T), a \in \mathcal{A}, a'_{i'j'} \in \{0, 1\}\}.$$

Therefore the only difference between A and A' is that $A_{i'j'} = 0$ in A but $A'_{i'j'}$ can be either 0 or 1 in A' . This also applies to the difference between an a and an a' . Following (3), we define the conditional precision matrix for \mathcal{S}' as

$$Q'(a') = Q(a) + \beta a'_{i'j'}(e_{i'} - e_{j'})(e_{i'} - e_{j'})^T.$$

For simplicity, we will often suppress a and a' in our notation below when there is no danger of confusion. We will write, for example, Q' and Σ for $Q'(a')$ and $\Sigma(a)$ respectively. Finally we define μ' for \mathcal{S}' accordingly based on Q' or, equivalently, the its inverse Σ' . It is important to point out that the marginal distribution for A under μ' is not the same as that under μ . In fact μ and μ' are related through the relation

$$\mu(A = a) = \mu'(A = a | A'_{i'j'} = 0).$$

We first establish a general relationship between $|\Sigma'|$ and $|\Sigma|$ (recall that the probability $\mu(A = a)$ is proportional to $|\Sigma|^{1/2}$). For all $i, j \in V$, let σ_{ij} and σ'_{ij} be the entries of Σ and Σ' respectively, the conditional covariance matrices of X under μ and μ' respectively, and let

$$\delta_{ij} = \sigma_{ii} + \sigma_{jj} - 2\sigma_{ij} \text{ and } \delta'_{ij} = \sigma'_{ii} + \sigma'_{jj} - 2\sigma'_{ij},$$

the conditional variances of $X_i - X_j$ under the distributions $N(0, \Sigma)$ and $N(0, \Sigma')$ respectively. Note that δ_{ij} and δ'_{ij} are functions of the network configurations a and a' respectively.

Lemma 1. *Let $a \in \mathcal{A}$ and $a' \in \mathcal{A}'$ be as related in (10). Then*

$$(11) \quad |\Sigma'| = (1 + \beta \delta'_{i'j'})^{-a'_{i'j'}} |\Sigma|,$$

and $\delta_{i'j'}$ and $\delta'_{i'j'}$ are related through

$$(12) \quad (1 - \beta \delta'_{i'j'})^{a'_{i'j'}} (1 + \beta \delta_{i'j'})^{a'_{i'j'}} = 1.$$

Proof. By definition

$$\Sigma' = (Q + \beta a'_{i'j'}(e_{i'} - e_{j'})(e_{i'} - e_{j'})^T)^{-1}.$$

From the Sylvester's identity for determinants

$$|I_n + UV| = |I_m + VU|,$$

which holds for any $n \times m$ matrix U and $m \times n$ matrix V , it follows that for any $n \times n$ invertible W ,

$$(13) \quad |W + UV| = |W| \cdot |I_n + (W^{-1}U)V| = |W| \cdot |I_m + VW^{-1}U|.$$

Now set $W = Q$, $U = \beta a'_{i'j'}(e_{i'} - e_{j'})$, and $V = (e_{i'} - e_{j'})^T$. Since

$$VW^{-1}U = \beta a'_{i'j'}(e_{i'} - e_{j'})^T \Sigma (e_{i'} - e_{j'}) = \beta a'_{i'j'} \delta_{i'j'},$$

(11) follows from (13) with $W + UV = Q'$. The identity (12) is based on the observation that $W + UV = Q'$ is nonsingular and

$$|W| = |(W + UV) - UV| = |W + UV| \times |I - V(W + UV)^{-1}U|$$

and a comparison of this equation to (13). □

To derive a formula for the conditional distribution of one edge given other edges, we formulate this problem in terms of the systems \mathcal{S} and \mathcal{S}' . Let $\delta'_{i'j'}$ be as defined above and further let

$$\delta_1(a) \equiv \delta'_{i'j'}(a' : a'_{ij} = a_{ij} \text{ for } (i, j) \in E \text{ and } a'_{i'j'} = 1)$$

and

$$\delta_0(a) \equiv \delta'_{i'j'}(a' : a'_{ij} = a_{ij} \text{ for } (i, j) \in E \text{ and } a'_{i'j'} = 0).$$

These are the conditional variances under μ' of the difference $X_i - X_j$ for a given a' in which $a'_{i'j'} = 1$ and $a'_{i'j'} = 0$ respectively. Note that in this notation, $\delta_0(a)$ and $\delta_{i'j'}(a)$ have the same value.

Proposition 2. *The conditional probability of $A_{i'j'}$ given $A = a$ under μ' , is*

$$(14) \quad \mu'(A_{i'j'} = 1 | A = a) = \frac{1}{1 + \sqrt{1 + \beta\delta_{i'j'}(a)}}$$

or, equivalently,

$$(15) \quad \mu'(A_{i'j'} = 1 | A = a) = \frac{\sqrt{\delta_1(a)}}{\sqrt{\delta_0(a)} + \sqrt{\delta_1(a)}}.$$

Proof. The first formula follows directly from (11) and (5) by noting that for a and a' as related in (10),

$$\mu'(A_{i'j'} = 1 | A_{ij} = a_{ij}, (i, j) \in E) = \frac{|\Sigma'(a' : a'_{i'j'} = 1)|^{1/2}}{|\Sigma'(a' : a'_{i'j'} = 0)|^{1/2} + |\Sigma'(a' : a'_{i'j'} = 1)|^{1/2}}$$

and canceling out from both the numerator and the denominator the common fraction $|\Sigma(a)|^{1/2}$.

To obtain the second formula, we note that with the new notation, (12) can be written as

$$(1 - \beta\delta_1(a))^{a'_{i'j'}} (1 + \beta\delta_0(a))^{a'_{i'j'}} = 1$$

which implies another interesting identity, when $a'_{i'j'} = 1$,

$$\frac{\delta_0(a)}{\delta_1(a)} = 1 + \beta\delta_0(a) = 1 + \beta\delta_{i'j'}(a).$$

Plugging this into (14), we obtain (15). □

This proposition asserts that, among other things, the conditional distribution of an edge A_{ij} in the graph depends on the rest of the edges through and only through the conditional standard deviations of $X_i - X_j$.

4. THE FKG INEQUALITY AND THE INFINITE-VOLUME WEAK LIMIT

In this section we study an asymptotic property of the edge distribution as the size of the graph approaches to infinity. We will first explore a property called positive association for our model which, together with the Proposition 2 in Section 3, will allow us to show the existence of a weak limit (convergence in distribution) on an infinite graph for this model. After Lemma 3 below, the properties for our graphical model are developed in parallel to some of those for the random-cluster model of [14].

We start with a relationship between the conditional covariance matrices Σ' and Σ given edges. It displays explicitly how adding one edge to the graph can affect the conditional covariances of the random variables in X .

Lemma 3. *For all a and a' as related in (10) and for any $i, j \in V$,*

$$(16) \quad \sigma'_{ij} = \sigma_{ij} - \frac{\beta a'_{ij'}}{1 + \beta \delta'_{ij'}} (\sigma_{ii'} - \sigma_{ij'}) (\sigma_{ji'} - \sigma_{jj'})$$

and

$$(17) \quad \delta'_{ij} = \delta_{ij} - \frac{\beta a'_{ij'}}{1 + \beta \delta'_{ij'}} (\sigma_{ii'} - \sigma_{ij'} - \sigma_{ji'} + \sigma_{jj'})^2.$$

Proof. Applying the matrix identity

$$(W + UV)^{-1} = W^{-1} - W^{-1}V(I + UW^{-1}V)^{-1}UW^{-1},$$

which holds for any valid (so that the inverses exist) and compatible (so that the products are defined) matrices, we have, as in the proof of the Proposition 1,

$$\Sigma' = \Sigma - \frac{\beta a'_{ij'}}{1 + \beta (e_{i'} - e_{j'})^T \Sigma (e_{i'} - e_{j'})} \Sigma (e_{i'} - e_{j'}) (e_{i'} - e_{j'})^T \Sigma.$$

Now (16) is just an element-wise version of this equation, and (17) is a direct consequence of (16) and the definitions of δ_{ij} and δ'_{ij} . \square

A special case of (16) is $i = j$. It implies that the conditional variance of X_i for each $i \in V$ become smaller when extra edges are added into the conditioning network or, in a more general form,

$$(18) \quad \text{Var}(X_i|a) \geq \text{Var}'(X_i|a'), \quad \forall i \in V,$$

whenever $a \leq a'$ in the sense that $a_{ij} \leq a'_{ij}$ for all $i, j \in V$. Similarly, (17) shows the conditional variance of $X_i - X_j$ is also decreasing in a :

$$(19) \quad \text{Var}(X_i - X_j|a) \geq \text{Var}'(X_i - X_j|a').$$

Hence a network with more edges has smaller conditional variances of X_i and $X_i - X_j$, for all $i, j \in V$.

We now show that our graphical model possesses a nice property called positive association which is characterized by the FKG inequality below ([14]). This inequality plays a fundamental role in studying some well-known models in statistical mechanics, including the Ising model and Potts model.

Proposition 4. *For any increasing functions f and g defined on $\{0, 1\}^E$, we have*

$$(20) \quad E_\mu(fg) \geq E_\mu(f)E_\mu(g).$$

Proof. For any $a, a' \in \{0, 1\}^E$, let

$$a \vee a' = \{\max\{a_{ij}, a'_{ij}\} \text{ for all } (i, j) \in E \text{ and } a_{ij} \in a, a'_{ij} \in a'\}$$

and

$$a \wedge a' = \{\min\{a_{ij}, a'_{ij}\} \text{ for all } (i, j) \in E \text{ and } a_{ij} \in a, a'_{ij} \in a'\}.$$

It is well-known that (20) is a consequence of the FKG lattice condition

$$(21) \quad \mu(a \vee a')\mu(a \wedge a') \geq \mu(a)\mu(a').$$

A statement and a proof of this result can be found in [14], page 25-26. According to again [14] (Theorem 2.24), the condition (21) is in turn equivalent to the “one-point conditional probability condition”. This later condition states that for any $(i, j) \in E$ and a given $a \in \{0, 1\}^{E - \{(i, j)\}}$, if $A^{-ij} \equiv \{A_{kl} : (k, l) \in E - \{(i, j)\}\}$, the conditional probability $\mu(A_{ij} = 1 | A^{-ij} = a)$ is increasing in a . To show this is true in our case, we note that according to (14) of Proposition 2, such a conditional probability is decreasing in the quantity $\delta_{ij}(a)$. Also (17) of Lemma 3 implies that for every $(i, j) \in E$, the function $a \rightarrow \delta_{ij}(a)$ is decreasing:

$$\delta_{ij}(a') \geq \delta_{ij}(a''), \text{ whenever } a' \leq a''.$$

It follows that the one-point conditional probability condition holds, and therefore (20) is true. \square

Finally we consider any sequence of finite graphs $G^{(n)} = (V^{(n)}, E^{(n)})$, $n = 1, 2, \dots$, such that $V^{(n)} \subset V^{(n+1)}$, $E^{(n)} \subset E^{(n+1)}$, and the size of $E^{(n)}$ tends to infinity. Let $A^{(n)}$ be the random

adjacency matrix defined on $G^{(n)}$ with the probability distribution $\mu_n \equiv \mu_{A^{(n)}}$ as given in (5):

$$(22) \quad \mu_n(a) = \frac{|\Sigma(a)|^{1/2}}{\sum_{a' \in \{0,1\}^{E^{(n)}}} |\Sigma(a')|^{1/2}}, \quad a \in \{0,1\}^{E^{(n)}}.$$

We are interested in the consistency property of the sequence of the distributions $\{\mu_n\}$ so that for any event B depending on some edges in a finite graph $G^{(n_0)}$, the probabilities $\mu_n(B)$, $n \geq n_0$, has a limit as $n \rightarrow \infty$. This problem can be formulated as follows.

Let

$$(23) \quad V = \lim V^{(n)} \text{ and } E = \lim E^{(n)},$$

the limits of the increasing sets. Let $\Omega = \{0,1\}^E$ and \mathcal{F} be the σ -field generated by the cylinder events of Ω . For each n , we can view μ_n as a probability measure defined on (Ω, \mathcal{F}) with a support on a subset

$$\Omega_n = \{0,1\}^{E^{(n)}} \times \{0\}^{E-E^{(n)}}$$

of Ω . The consistency problem is then the problem of weak convergence of the sequence $\{\mu_n\}$ on (Ω, \mathcal{F}) .

Let \mathcal{F}_n be the σ -field generated by the subsets of Ω_n . Then $\mathcal{F}_n \subset \mathcal{F}_{n+1}$. We notice that for any $m < n$ and $B_1 \in \mathcal{F}_m$, if $B_2 \in \mathcal{F}_n$ is the event that $A_{ij} = 0$ for all edges in $E^{(n)} - E^{(m)}$: $B_2 = (A_{ij} = 0, \forall (i,j) \in E^{(n)} - E^{(m)})$, then according to (22) and (3),

$$(24) \quad \mu_m(B_1) = \mu_n(B_1|B_2).$$

We will say an edge event B is increasing if the corresponding indicator function $I_B(a)$ is increasing. If B is decreasing then the negative of its indicator function is increasing.

Corollary 5. *For any $m < n$ and any increasing event $B_1 \in \mathcal{F}_m$,*

$$(25) \quad \mu_m(B_1) \leq \mu_n(B_1).$$

Proof. Let B_2 be as defined above. Then B_2 is a decreasing event and $\mu_n(B_2) > 0$. The FKG inequality (20) in this case implies

$$\mu_n(B_1 \cap B_2) \leq \mu_n(B_1)\mu_n(B_2),$$

or

$$\mu_n(B_1|B_2) \leq \mu_n(B_1).$$

(25) follows from this and (24). \square

Now we have:

Proposition 6. *As $n \rightarrow \infty$, $A^{(n)}$ converges in distribution to a random adjacency matrix A of the graph G in the probability space $(\Omega, \mathcal{F}, \nu)$ for some probability distribution ν .*

Proof. The statement in the proposition is equivalent to asserting the weak convergence of μ_n to some ν on (Ω, \mathcal{F}) or, equivalently, to the statement that there is a ν on (Ω, \mathcal{F}) such that for every finite dimensional cylinder event $B \in \mathcal{F}_n \subset \mathcal{F}$,

$$(26) \quad \lim_n \mu_n(B) = \nu(B).$$

Note that B is discrete with $\partial B = \emptyset$ in the discrete topology and therefore is always ν -continuous ($\nu(\partial B) = 0$). To establish (26), we follow an argument of Grimmett in proving his Theorem 4.19 (a) in [14] by first assuming that B is an increasing event. The Corollary 5 then states that

$$\mu_n(B) \leq \mu_{n+1}(B).$$

Therefore the limit (26) holds for all increasing events B . Since the set of increasing events forms a convergence-determining class (Billingsley [2]), (26) must hold for all the subsets in \mathcal{F} for a probability measure ν on (Ω, \mathcal{F}) . \square

We end this section with two more observations.

First, suppose $\mathcal{S}' = (V', E', A', X', \mu')$ and $\mathcal{S}'' = (V'', E'', A'', X'', \mu'')$ be two finite systems defined as before such that $V' \subset V''$, $E' \subset E''$. Because of the monotonic properties of the functions $a \rightarrow \text{Var}(X_i|a)$ and $a \rightarrow \delta_{ij}(a)$ implied in (18) and (19), Corollary 5 implies that for all $i, j \in V$, the variances of X_i and $X_i - X_j$, as functions of the edge set E , are decreasing in the sense that

$$\text{Var}'(X_i) \geq \text{Var}''(X_i), \text{Var}'(X_i - X_j) \geq \text{Var}''(X_i - X_j).$$

Finally, the formula (16) demonstrates exactly how Σ' depends on $A_{i'j'}$. This observation leads to a martingale representation for the quantity $\log |\Sigma(A)|$ as follows. To state the result, let us label all edges in the graph in some (arbitrary) order so that we can write $E = \{(i_1, j_1), \dots, (i_k, j_k)\}$.

Proposition 7. *Let $A = \{A_{ij}\}_{i,j \in V}$ be any symmetric random adjacency matrix from the distribution μ_A . For $k = 1, \dots, n$, let*

$$A^{(k)} = \sum_{l=1}^k A_{i_l j_l} (e_{i_l} - e_{j_l})(e_{i_l} - e_{j_l})^T$$

and $\mathcal{F}_k = \sigma\{A_{i_1 j_1}, \dots, A_{i_k j_k}\}$. Then $\{-\log |\Sigma(A^{(k)})|, \mathcal{F}_k, k = 1, \dots, n\}$ forms a sub-martingale.

Proof. Applying (11) repeatedly, we can write, for $k = 1, \dots, n$,

$$-\log |\Sigma(A^{(k)})| = \sum_{l=1}^k A_{i_l j_l} \log \left[1 + \beta \delta_{i_l j_l} (A^{(l-1)}) \right],$$

where $A^{(0)}$ is defined to be the 0 matrix. The Proposition follows by noting that $\delta_{i_l j_l} (A^{(l-1)}) \in \mathcal{F}_{l-1}$ for $l = 1, \dots, n$ (with \mathcal{F}_0 being the trivial σ -field) and all the terms in the summation above are positive. \square

REFERENCES

- [1] Bickel, P.J., Chen, A., 2009. A nonparametric view of network models and Newman-Girvan and other modularities, *Proceedings of the National Academy of Sciences (USA)*, 106, 21068 - 21073.
- [2] Billingsley, P., 1968. *Convergence of Probability Measures*, Wiley, New York.
- [3] Bollobas, B., 2001. *Random Graphs*, 2nd ed., Cambridge University Press, New York.
- [4] Borgs, C., Chayes, J.T., Lovász, L., Soth, V.T., and Vesztergombi, K., 2008, Convergent graph sequences I: Subgraph frequencies, metric properties, and testing, *Advances in Math.* 219, 1801–1851.
- [5] Borgs, C., Chayes, J.T., Lovász, L., Soth, V.T., and Vesztergombi, K., 2012, Convergent graph sequences II: Multiway Cuts and Statistical Physics, *Annals of Math.* 176, 151–219.
- [6] Chung, F. and Lu, L., 2006, *Complex Graphs and Networks*, American Mathematical Society, Providence.
- [7] Diaconis, P. and Janson, S., 2008. Graph limits and exchangeable random graphs. *Rend. Mat. Appl.* (7) 28 33–61.
- [8] Durrett, R., 2010. *Random Graph Dynamics*, Cambridge University Press, New York.
- [9] Handcock, M.S. and Raftery, A.E., 2007, Model-based clustering for social networks, *J. R. Statist. Soc. A*, 170, 301-354.

- [10] Hans-Otto Georgii, 1988, Gibbs Measures and Phase Transitions, Walter de Gruyter, Berlin New York.
- [11] Hoff, P. D., Raftery, A. E. and Handcock, M. S., 2002. Latent space approaches to social network analysis. J. Am. Statist. Ass., 97, 1090–1098.
- [12] Holley, R., 1974, Remarks on the FKG inequalities, Communications in Mathematical Physics, 36, 227-231.
- [13] Goldenberg, A., Zheng, A.X. , Fienberg, S.E., Airoldi, E.M., 2010. A Survey of Statistical Network Models, Foundations and Trends in Machine Learning, v.2 n.2, p.129-233.
- [14] Grimmett, R., 2006. The Random-Cluster Model, Springer-Verlag Berlin Heidelberg.
- [15] Kolaczyk, E.D., 2009. Statistical Analysis of Network Data: Methods and Models, Springer Science+Business Media LLC.
- [16] Krivitsky, M.S. and Handcock, A., 2013. A separable model for dynamic networks. J. Roy. Stat. Soc. Ser. B (Stat. Meth.).
- [17] Lovász, L., 2012 Large networks and graph limits, Amer. Math. Soc., Providence, R.I..
- [18] McAuley, J. J., and Leskovec, J., 2012. Learning to discover social circles in ego networks. Proc. NIPS (2012).
- [19] Mislove, A., Viswanath, B., Gummadi, K. and Druschel P., 2010. You are who you know: Inferring user profiles in online social networks. In WSDM, 2010.
- [20] Newman, M., 2010. Networks: An Introduction, Oxford University Press, Oxford.
- [21] Saijders, T.A., Van de Bunt, G.G. and Steglich, C.E., 2010. Introduction to stochastic actor-based models for network dynamics, Soc. Network, 32(1), 44-66.
- [22] Wang, Y.X. and Haiyan Huang, 2014, Review on statistical methods for gene network reconstruction using expression data, Journal of Theoretical Biology, 362, 53-61.

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF MISSOURI - ST. LOUIS, ST. LOUIS, MO 63121.

E-mail address: haiyan_cai@umsl.edu